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A new Hilbert space approach to the multimode squeezing of light

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Abstract. The Hilbert space of one-photon states is introduced and used for an investigation of some basic principles of squeezing. Regarding the one-photon space as the underlying structure of the fields, we introduce parametrisations that simplify calculations considerably, and discuss how phase sensitivity, characteristic for squeezing, reveals itself in the one-photon space.

1. Introduction

The quantum theory of light has always drawn benefit from using descriptions that, besides fulfilling the principles of quantum mechanics, also bear a close resemblance to classical pictures. Perhaps the most well known example is that of coherent states (Glauber 1963), reproducing classical behaviour in the limit of strong fields. Other examples with increasing importance for the description of amplification processes (Stenholm 1986) are the various characteristic and distribution functions, as they are often compared with probability functions on a classical phase space (their properties may deviate considerably from classical probability functions).

A state or a process is termed non-classical if some conditions are violated that would have been obeyed by a classical field. This violation can often be ascribed to the commutator $[a_k, a_k^+] = I$ (a_k and a_k^+ are the operators that annihilate and create a quantum of energy $\hbar\omega_k$ in the electromagnetic field mode labelled by k), and therefore stems from the discreteness—or the particle nature—of the light field.

In spite of this observation little attention is paid to the importance of the one-photon states as 'generators' of the complete set of field states in quantum optics. In this way quantum optics differs from the mathematical description of quantum fields where the one-particle states play a prominent role (Bargmann 1961).

The one-particle state—the photon—is the fundamental basis for the non-classical behaviour of light, and in this paper we shall construct the field theory from the one-photon states. This will be done in § 2 and is of course a not very complicated task since it already exists, and § 2 may therefore be seen as a presentation of the notation and the way of thinking for this description. At the end of the section we introduce the Bargmann representation, which is intimately connected to the point of departure taken here, and also provides a unique means of calculation.

In § 3 we consider the squeezed states of light and use the previous section to give some simple derivations and interpretations of importance for squeezing. Calculus will be kept to a minimum, and for more detailed calculations and mathematical rigour we refer to Slowikowski and Mølmer (1988, hereafter referred to as I), where some of the more mathematical concepts are discussed.

2. Construction of field theory

We shall regard one-photon states in uncorrelated modes as orthogonal unit vectors in a Hilbert space H with inner product $\langle\ |\ \rangle$. Denoting the one-photon state in mode m by e_m, $\{e_m\}$ is an orthonormal basis of H, in which of course only linear combinations that are unit vectors bear immediate physical significance as states of one photon.

We now define product states, states of more than one photon. This demands for a definition of multiplication and a multiplicative unit, the vacuum, which we shall denote by $\varnothing$. We shall just write xy for the product of states x and y, and we define the space of all polynomials

$$\mathscr{F}_0=\left\{t\varnothing+\sum a_1a_2\ldots a_j \,\middle|\, a_j\in H,\ t\in\mathbb{C}\right\}.$$

With a_j, b in H we extend the definition of $\langle\ |\ \rangle$ by

$$\begin{aligned}&\langle\varnothing|\varnothing\rangle=1\\ &\langle a_1a_2\ldots a_j\,|\,b^k\rangle=\delta_{j,k}k!\langle a_1|b\rangle\langle a_2|b\rangle\ldots\langle a_j|b\rangle.\end{aligned}\tag{2.1}$$

The completion of $\mathscr{F}_0$ with respect to $\langle\ |\ \rangle$ is just the usual Fock space, $\mathscr{F}$, which is also a Hilbert space, where again only the unit vectors, $|f|=\langle f\,|\,f\rangle^{1/2}=1$, represent physical states.

Of special importance are the coherent states

$$\exp(-\tfrac{1}{2}|x|^2)\exp(x)=\exp(-\tfrac{1}{2}|x|^2)\sum n!^{-1}x^n \qquad x\in H$$

where $x^0=\varnothing$ and the factor of normalisation is found from (2.1) and the resulting $\langle\exp(x)\,|\exp(y)\rangle=\exp(\langle x\,|\,y\rangle)$.

Useful algebraic properties of the coherent states follow from $\exp(x)\exp(y)=\exp(x+y)$, x, $y\in H$.

If f and g are vectors in $\mathscr{F}_0$, so is fg. More generally, to any state f in $\mathscr{F}$ we assign the creation operator $a^+(f)$ through its action $a^+(f)g=fg$ on g for which $fg\in\mathscr{F}$. When it exists the adjoint of $a^+(f)$ is denoted by $a(f)$ and is called annihilation by f.

The usual creation operators of one quantum of energy, a_m^+, are just one example (f being e_m), and if f is some specific polynomial in the e_m, then $a^+(f)$ is given by the same polynomial in the $a^+(e_m)$. A wide class of operators may be written as functions of the one-photon creation and annihilation operators, and may then be ordered in different ways using $[a(x), a^+(y)]=\langle x\,|\,y\rangle I$, x, $y\in H$. One example is the displacement operator

$$D(z)=\exp(a^+(z)-a(z))=\exp(-\tfrac{1}{2}|z|^2)a^+(\exp(z))a(\exp(-z)) \qquad z\in H \tag{2.2}$$

a unitary operator that, acting upon the vacuum $\varnothing$, gives the normalised coherent states $\exp(-\frac{1}{2}|z|^2)\exp(z)$.

Another class of operators are the number operators $n_m=a^+(e_m)a(e_m)$, the expectation values of which give the average number of photons in the mth mode. Eigenstates of n_m are states with a specific number of photons in the given mode.

Notice that the construction above in retrospect can be formulated as follows. In the space of field states we define a special set of vectors $\{e_m\}$, each e_m being an eigenvector for all number operators $n_ke_m=\delta_{k,m}e_m$. With the usual scalar product these vectors define a Hilbert space $H=\{\Sigma\beta_ie_i\,|\,\beta_i\in\mathbb{C}\}$ of eigenvectors for the total number operator $N=\Sigma\, n_m$. The eigenvalue is 1, and the unit vectors therefore describe one-photon states.

The coherent states give an overcomplete basis for the field states, and any state f will therefore be fully determined through its scalar product with all of the coherent states, or equivalently $\langle \exp(z) | f \rangle$, $z \in H$. Via the identification $z = \Sigma\, \beta_i e_i$ $\beta_i \in \mathbb{C}$, one gets in the case of a finite number of modes M that, to each state of the field, $\langle \exp(z) | f \rangle$ defines a function $f_B : \mathbb{C}^M \to \mathbb{C}$. This is called the Bargmann representation (I E Segal 1960 (see Feller 1962), Bargmann 1961, Glauber 1963) and comprises several useful analytical properties.

As a pure consequence of the definition of $\langle\ |\ \rangle$ in $\mathscr{F}$, but of fundamental importance for the representation, is

$$\langle \exp(z) | fg \rangle = \langle \exp(z) | f \rangle \langle \exp(z) | g \rangle. \tag{2.3}$$

This is easily seen from (2.1) and thus says that the Bargmann representation is multiplicative: $(fg)_B = f_B g_B$. The functions f_B also define a Hilbert space, because the Gaussian measure $\mathrm{d}\mu = \pi^{-M} \exp(-\Sigma\, |\beta_i|^2)\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2 \ldots \mathrm{d}\beta_M$ allows for the scalar product $\int_{\mathbb{C}^M} \mathrm{d}\mu\, f_B^* g_B$ that even makes the Bargmann representation isometric: $\langle f | g \rangle = \int_{\mathbb{C}^M} \mathrm{d}\mu\, f_B^* g_B$.

If f_B is some specific polynomial or power series in β_i then f is obtained by substituting the one-photon states e_i for β_i. The Bargmann representation is thus very well suited for the approach outlined at the start of this section. Finally, we mention that the Bargmann representation directly gives the expansion of a state upon the coherent states

$$f = \int_{\mathbb{C}^M} \mathrm{d}\mu\, f_B \exp(z). \tag{2.4}$$

3. Squeezed states of light

Now we shall address ourselves to a multimode description of squeezed light (for an extensive overview of aspects and applications of squeezed light, see Walls and Kimble (1987)).

First we will deal with some operator relations of importance for squeezing. We shall first parametrise the squeezed vacuum states, and later the unitary squeeze operator, by a special class of operators on the one-photon space, H. These operators allow for a kind of separation, implying a simple generalisation of the single-mode results, with which we shall make some comparison. Apart from generalisations of single-mode results, we have derived the normal order form of the squeeze operator and the effect of squeezing upon a coherent state.

A transformation of operators reveals systematic properties that we shall use to derive the infinitesimal generator of squeezing. An essential condition for squeezing is phase sensitivity. This lies implicit in our description, and we shall try to extract it at the end of the section.

Consider operators $L : H \to H$ of the type

$$Lx = \sum t_m \langle x | e_m \rangle e_m \tag{3.1}$$

where $t_m \in \mathbb{C}$ and $\sup|t_m| < 1$ $(\Sigma\, |t_m|^2 < \infty$ in the case of infinitely many modes).

To each L we now assign $\hbar_L = \Sigma\, e_m(Le_m)$ and

$$\delta_L = \exp(-\tfrac{1}{2}\hbar_L) = \Sigma\, n!^{-1}(-\tfrac{1}{2}\hbar_L)^n \in \mathscr{F}. \tag{3.2}$$

$\det(I-L^2)^{1/4}\delta_L$ is a unit vector (Kristensen *et al* 1967) and is, since $|\not{k}_L|^{-1}\not{k}_L$ is a two-photon state, called a two-photon coherent state, or a squeezed vacuum state. Since $\not{k}_{M+N}=\not{k}_M+\not{k}_N$, we get $\delta_{M+N}=\delta_M\delta_N$, where M, N, $M+N$ are of the type specified above in analogy with the similar result for the 'one-photon' coherent states.

In I we derive the relation

$$a(\delta_L)a^+(\exp(x))=\exp(-\tfrac{1}{2}\langle Lx\,|\,x\rangle)a^+(\exp(x))a(\exp(-Lx))a(\delta_L). \tag{3.3}$$

This provides one of the means for doing calculations with the annihilation operator $a(\delta_L)$. As a direct result the vacuum expectation values of both sides imply

$$\langle\exp(z)\,|\,\delta_L\rangle=\exp(-\tfrac{1}{2}\langle z\,|\,Lz\rangle)$$

and thus give the coherent-state expansion (2.4) of the squeezed vacuum states; see also (3.6). It is easy to show that δ_L and $\not{k}_L$ are independent of the choice of basis (note that L is conjugate linear, x is to the left in the scalar product in (3.1)), but the formal description is facilitated and the mathematical structure becomes more prominent using a basis as in (3.1) with $t_n\in\mathbb{R}$. Within this basis of eigenmodes $\not{k}_L$ is a quadratic form without cross terms, and the multimode description is a simple generalisation of the single-mode case. Our notation therefore differs from the usual one (Caves 1982), which associates the field with a carrier frequency, Ω, and 'squeezes together' pairs of modes, $e_{\Omega+\varepsilon}$, $e_{\Omega-\varepsilon}$. The corresponding operators are easily translated into our notation.

Defining $e_{\varepsilon\pm}=2^{-1/2}(e_{\Omega+\varepsilon}\pm e_{\Omega-\varepsilon})$, one gets $a^+(e_{\Omega+\varepsilon})a^+(e_{\Omega-\varepsilon})=\frac{1}{2}(a^{+2}(e_{\varepsilon+})-a^{+2}(e_{\varepsilon-}))$, the factor $\frac{1}{2}$ playing the same role here as when multiplying $\not{k}_L$ in (3.2).

Yao (1987) gives the following expression for the single-mode squeeze operator:

$$S(\xi)=\exp[\tfrac{1}{2}(\xi a^{+2}-\xi^* a^2)]=(\operatorname{sech} r)^{1/2}\exp(\tfrac{1}{2}\tanh r\,\mathrm{e}^{2\mathrm{i}\theta}a^{+2})$$
$$\times\exp(-n\ln(\cosh r))\exp(-\tfrac{1}{2}\tanh r\,\mathrm{e}^{-2\mathrm{i}\theta}a^2).$$

Replacing

$$\xi=r\,\mathrm{e}^{2\mathrm{i}\theta}=\tfrac{1}{2}\ln\left(\frac{1-t}{1+t}\right)\mathrm{e}^{2\mathrm{i}\theta}$$

(t real, $|t|<1$), this turns into

$$S_t=(1-t^2)^{1/4}\exp(-\tfrac{1}{2}t\,\mathrm{e}^{2\mathrm{i}\theta}a^{+2})\exp(n\ln(1-t^2)^{1/2})\exp(\tfrac{1}{2}t\,\mathrm{e}^{-2\mathrm{i}\theta}a^2)$$

and, by acting upon vacuum, we immediately get the single-mode version of $\delta_L/|\delta_L|$, hereby confirmed to represent the squeezed vacuum vectors. The phase $\mathrm{e}^{2\mathrm{i}\theta}$ is absorbed by a redefinition of the mode, $e_m\to\mathrm{e}^{\mathrm{i}\theta}e_m$.

It also justifies the following suggestion for the unitary multimode squeeze operator:

$$S_L=\det(I-L^2)^{1/4}a^+(\delta_L)\exp\left(\sum\ln(1-t_m^2)^{1/2}a^+(e_m)a(e_m)\right)a(\delta_{-L}). \tag{3.4}$$

In the mathematical framework of I (3.4) appears in a natural manner; in particular the exp term is an appropriate extension of the linear operator $(I-L^2)^{1/2}$: $H\to H$ into an operator $\mathscr{F}\to\mathscr{F}$.

The normal-order form of (3.4) is readily obtained by using the result (the single-mode version of which was used in (Fan *et al* 1987))

$$\exp\sum\omega_m a^+(e_m)a(e_m)={:}\exp\sum(\exp(\omega_m)-1)a^+(e_m)a(e_m){:}$$

by which

$$\exp\left(\sum \ln(1-t_m^2)^{1/2} a^+(e_m)a(e_m)\right)$$
$$= :\exp \sum_m [\exp(\ln(1-t_m^2)^{1/2})1] a^+(e_m)a(e_m):$$
$$= :\sum_n n!^{-1}\left(\sum_m [(1-t_m^2)^{1/2}-1]a^+(e_m)a(e_m)\right)^n :$$
$$= \sum_n \sum_j \prod_m (j_m!)^{-1}[(1-t_m^2)^{1/2}-1]^{j_m} a^{+j_m}(e_m)a^{j_m}(e_m) \tag{3.5}$$

where the sum Σ_j is to be taken over all combinations of j_m with $\Sigma_m j_m = n$. Insertion of (3.5) into (3.4) gives the squeeze operator in normal form. Higher-order correlation functions in general appear as expectation values of operators like those in (3.5), but it is at present unclear whether any direct physical interpretation can be given to (3.5).

We have hitherto only discussed the squeezed vacuum vectors $S_L\varnothing = \det(I-L^2)^{1/4}\delta_L$. Displacement by (2.2) does not change the noise properties, and in the literature the term 'squeezed vector' embraces all states obtained by first acting with S_L upon $\varnothing$ followed by the displacement operator $D(z)$. Repeatedly used, (2.3) and (3.3) give

$$\langle \exp(z) \,|\, D(x)S_L\varnothing\rangle$$
$$= \exp(-\tfrac{1}{2}|x|^2)\det(I-L^2)^{1/4}\langle \exp(z)|a^+(\exp(x))a(\exp(-x))a^+(\delta_L)|\varnothing\rangle$$
$$= \det(I-L^2)^{1/4}\exp(-\tfrac{1}{2}|x|^2+\langle z\,|\,x\rangle-\tfrac{1}{2}\langle x\,|\,Lx\rangle-\tfrac{1}{2}\langle z\,|\,Lz\rangle+\langle z\,|\,Lx\rangle). \tag{3.6}$$

Now (2.4) gives explicitly the coherent-state representation of a squeezed state. In the single-mode case this result, by the identification $x=\alpha e_1$, $z=\beta e_1$, $|\beta\rangle = \exp(-\tfrac{1}{2}|\beta|^2)|\exp(\beta e_1)\rangle$, reduces exactly to the expansion found by Yao (1987).

In I we derive the action of S_L upon a coherent state

$$S_L D(x)\varnothing = D([I-L)(I+L)^{-1}]^{1/2}x)S_L\varnothing. \tag{3.7}$$

This is also a standard squeezed state.

If a common basis of 'eigenmodes' exists for M, N, and if M, N and $M+N$ are all of the same type as L in (3.1), then

$$S_M S_N = S_{(M+N)(I+MN)^{-1}}. \tag{3.8}$$

Both of these results suggest using the operator transform

$$\kappa(L) = (I-L)(I+L)^{-1}. \tag{3.9}$$

Noting that $\kappa(AB) = (\kappa(A)+\kappa(B))(I+\kappa(A)\kappa(B))^{-1}$ and $\kappa(\kappa(L)) = L$, we define $A = \kappa(M)$, $B = \kappa(N)$ and thus rewrite (3.7),

$$S_L D(x)\varnothing = D(\kappa(L)^{1/2}x)S_L\varnothing$$

and (3.8)

$$S_{\kappa(A)}S_{\kappa(B)} = S_{\kappa(AB)}.$$

The first use of the new transform will be to calculate the infinitesimal generator for squeezing.

If L is given by (3.1), $A = \kappa(L)$ can be written

$$Ax = \sum \{r_m \rho(e_m, x)e_m + r_m^{-1}\rho(\mathrm{i}e_m, x)\mathrm{i}e_m\} \tag{3.10}$$

where $r_m = (1-t_m)/(1+t_m)$ and $\rho(\ ,\ )$ is the real scalar product $\rho(x, y) = \frac{1}{2}(\langle x|y\rangle + \langle y|x\rangle)$.

The infinitesimal generator of squeezing is now found as follows. We define

$$A^\tau x = \sum r_m^\tau \rho(e_m, x)e_m + r_m^{-\tau}\rho(\mathrm{i}e_m, x)\mathrm{i}e_m \qquad \tau > 0.$$

$A^{\tau+\nu} = A^\tau A^\nu$, and we see that $S_{\kappa(A^\tau)}$ defines a unitary group, the generator being the derivative that after some calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau} S_{\kappa(A^\tau)}|_{\tau=0} = \tfrac{1}{4}\sum \ln r_m(a^+(e_m^2) - a(e_m^2)).$$

This is the Hamiltonian that generates a squeezed state, and we get immediately with $\tau = 1$

$$S_L = \exp\left(\tfrac{1}{4}\sum \ln[(1-t_m)/(1+t_m)](a^+(e_m^2) - a(e_m^2))\right)$$

which is just the expected generalisation of the single-mode result.

The above calculation was not the only purpose for introducing the operator transformation and the real scalar product $\rho(\ ,\ )$. They both reveal important features of the conjugate linear operator L and of squeezing as seen from the one-photon viewpoint. The one-photon space, H, is a complex Hilbert space. Due to the mode separation (3.1) we can, in this context, look at one mode at a time, a subspace isomorphic to the complex plane. States of the field are then, through the Bargmann representation, isomorphic to conjugate analytic functions on the complex plane—even mixed states may be dealt with, representing the density matrix by distribution functions.

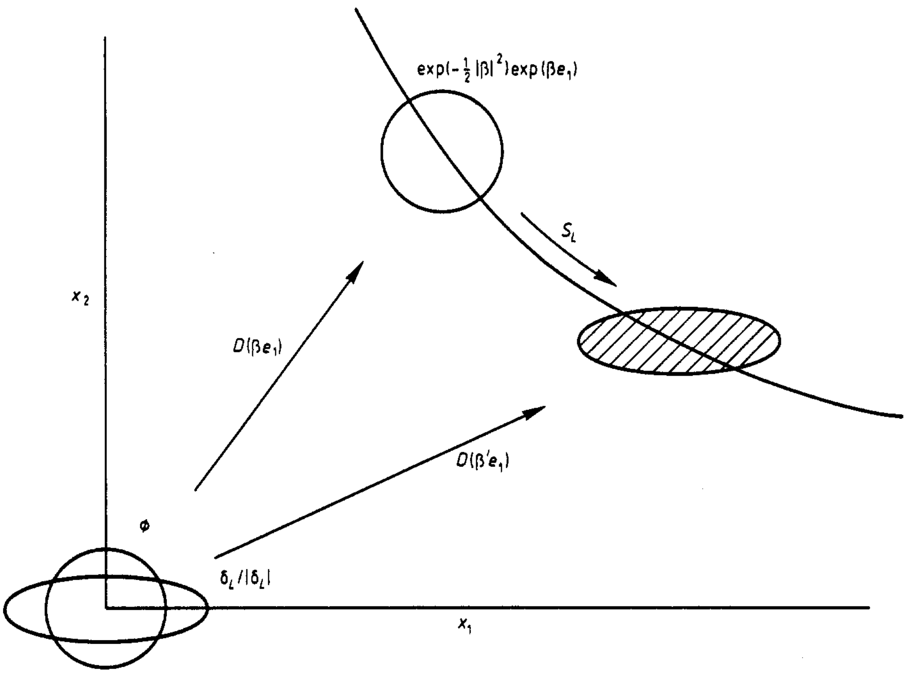

Figure 1. Uncertainty ellipses for a single-mode field in various states. $\exp(-\frac{1}{2}|\beta|^2)\exp(\beta e_1)$ is a coherent state, $\varnothing$ is the vacuum state, $\delta_L/|\delta_L|$ is the squeezed vacuum state, and the hatched area denotes a squeezed coherent state (see explanation in the text).

The definition of what is real and imaginary in H is now connected to physical distinctions, for example between position and momentum in the harmonic oscillator case, or in-phase and in-quadrature components of the electromagnetic field. Since $\rho(x, \mathrm{i}x)=0$ for all x in H, the real scalar product exactly provides the required phase sensitivity, and appears naturally when one is dealing with operators exhibiting special conjugation properties. In (3.10) A acts as a linear operator on the real vector space spanned by the basis $\{e_n, \mathrm{i}e_m\}$, compensating a scaling of the real projection along e_m by the reciprocal scaling of the projection along $\mathrm{i}e_m$, a squeezing.

In the single-mode case we can visualise relation (3.7) by uncertainty ellipses in the complex plane (see figure 1). By acting upon $\exp(-\frac{1}{2}|\beta|^2)\exp(\beta e_1)$, $\beta=b_1+\mathrm{i}b_2$, S_L modifies the field quadrature variances by the factors r, respectively r^{-1}, of (3.10), and the state obtained is equal to the result of squeezing the vacuum state by S_L and then displacing it by $D(\beta' e_1)$, $\beta'=r^{1/2}b_1+\mathrm{i}r^{-1/2}b_2$. The uncertainty ellipses of states obtained from real squeezing parameters (t in (3.1)) are all centred on the hyperbola going through β. A rotation of the coordinate axes, equivalent to the use of a complex t value, gives a completely different set of hyperbolae, and since the uncertainty ellipses now orient along the rotated coordinate axes, the uncertainty product is obviously increased in the original coordinate system.

4. Conclusion

The theory of squeezed states of light has been reviewed from the extreme non-classical viewpoint, the one-photon states. The pertaining product-state description offers several computational advances, and relations of importance for squeezing have been derived in a straightforward manner. Working within the one-photon space, basis transformations and phase-sensitive scaling operations have proved useful at calculating and visualising the effect of squeezing.

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